Topology Final Examination, 2010, B.Math 3rd Yr

(Total 60 mks. Attempt all questions. All questions carry equal (=12) marks. Any result proved in class maybe cited and used without proof to make your answers as brief as possible. Notes and books maybe consulted.)

1. (i): Show that the first uncountable ordinal space S_Ω is not homeomorphic to the space $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers with its usual topology.

   (ii): Prove that the function $\tau : S_\Omega \to S_\Omega$ defined by :
   $$\tau(x) = x_+ := \text{smallest}\{z \,:\, z > x\}$$
   is not continuous. (*Hint:* Consider what happens at the first infinite ordinal)

2. (i): Show that the space $\mathbb{R}_l$ is totally disconnected (i.e. its connected components are singletons. Here $\mathbb{R}_l$ denotes $\mathbb{R}$ with the lower limit topology)

   (ii): Prove that $\mathbb{R}_l \times \mathbb{R}_l$ (with the product topology) is not metrisable. .

3. (i): Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function with no zeros. Let $n \neq 0$ be a fixed integer. Prove that there exists a continuous function $h : \mathbb{C} \to \mathbb{C}$ such that $f(x) = h(x)^n$ for all $x \in \mathbb{C}$.

   (ii): Prove that there exists no continuous function $h : \mathbb{C} \to \mathbb{C}$ satisfying $h(x)^3 = x$ for all $x \in \mathbb{C}$. (Contrast this with the case of $\mathbb{R}$, where the continuous cube-root function $h(x) = x^{1/3}$ satisfies $h(x)^3 = x$ for all $x \in \mathbb{R}$).

4. (i): Let X, Y and Z be path-connected, locally path-connected and semilocally simply-connected spaces. Further assume that Z is simply connected and $\widetilde{X}$ (the simply-connected cover of X) is contractible. Show that every continuous map $f : Z \to X$ is nullhomotopic.

   (ii): Let $f : S^2 \to \mathbb{RP}^2$ be a continuous map. Show that if f is not nullhomotopic, then card $f^{-1}(x) \geq 2$ for all $x \in \mathbb{RP}^2$. (The natural quotient map $\pi : S^2 \to \mathbb{RP}^2$ shows that this inequality is sharp).

5. (i): Compute the fundamental group of $S^1 \vee S^2$.

   (ii): Let $X = U \cup V$ with U, V and $U \cap V \neq \phi$ all open and path connected, and let $x_0 \in U \cap V$. Suppose U is simply-connected. Show that $\pi_1(X, x_0)$ is a quotient of $\pi_1(V, x_0)$ and describe its kernel.